A DIRECTIONAL NOTION OF MULTIVARIATE EXTREME VALUE ANALYSIS

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Introduction and Motivation

1. **Extreme Value Theory concerns to the limit behavior of the sample extremes, (max or min in the univariate case)**

BUT

2. **Multivariate analysis is mandatory because Extremes are generated by many variables acting jointly with different relationships**
   - Asymptotic Independence & Asymptotic Dependence, (Pairs relations).
   - Correlations, (Overall relation).

Directional Perspective

1. **Look at the data with different perspectives to improve the identification and visualization of extremes**

BUT

- There are infinite directions to analyze the data, how to select an interesting one?.

1. **The classical tool for Extremes identification is the $\alpha-$quantile concept**

BUT

- There is a lack of a total order in $\mathbb{R}^d$.
- Conditioned to the $\alpha-$level, there are 2 approaches of estimation, $In - Sample$ and $Out - Sample$.

Importance of α in the estimation

In-Sample Vs. **Out-Sample**

In-Sample	Out-Sample
$\alpha > \frac{1}{n}$	$\alpha \leq \frac{1}{n}$
Some Observations Available	**Non-Observations Available**
↓	↓
Standard Estimation Procedures	**Multivariate Extreme Value Theory**

Outline

$$\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} \equiv$$ **QR Oriented Orthant. (Torres et al. 2015)**

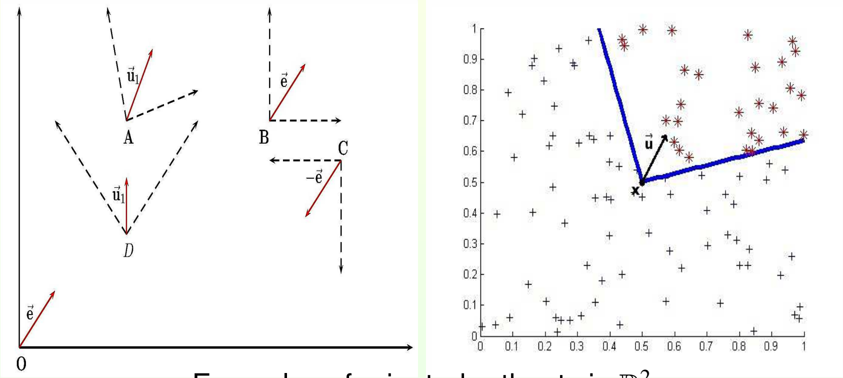
Definition

Given $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{u} \in \left\{\mathbf{z} \in \mathbb{R}^d : ||\mathbf{u}|| = 1, u_i \neq 0 \text{ for all } i = 1, ..., d\right\}$, the QR oriented orthant with vertex $\mathbf{x}$ and direction $\mathbf{u}$ is:

$$\mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} = \{\mathbf{z} \in \mathbb{R}^d | R_{\mathbf{u}}(\mathbf{z} - \mathbf{x}) \geq 0\},$$

where $\mathbf{e} = \frac{1}{\sqrt{d}}(1, ..., 1)'$ and $R_{\mathbf{u}}$ is an unique orthogonal matrix obtained by a QR decomposition, such that $R_{\mathbf{u}}\mathbf{u} = \mathbf{e}$.

Examples of Oriented Orthants



Examples of oriented orthants in $\mathbb{R}^2$

$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) \quad \equiv \quad$ **Directional Multivariate Quantile (Laniado et al. 2012)**

Definition

Given $\mathbf{u} \in \mathbb{R}^d, \; ||\mathbf{u}|| = 1$ and a random vector $\mathbf{X}$ with distribution probability $\mathbb{P}$, the α-quantile curve in direction $\mathbf{u}$ is defined as:

$$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) := \partial\{\mathbf{x} \in \mathbb{R}^d : \mathbb{P}(\mathfrak{C}_{\mathbf{x}}^{-\mathbf{u}}) \geq 1 - \alpha\},$$

where ∂ means the boundary and $0 \leq \alpha \leq 1$

$$\mathcal{U}_{\mathbf{X}}(\alpha, \mathbf{u}) \quad \equiv \quad \textbf{Directional Multivariate Upper Level-Set}$$

$$\mathcal{L}_{\mathbf{X}}(\alpha, \mathbf{u}) \quad \equiv \quad \textbf{Directional Multivariate Lower Level-Set}$$

Definition

Those sets are defined by:

$$\mathcal{U}_{\mathbf{X}}(\alpha, \mathbf{u}) := \{\mathbf{x} \in \mathbb{R}^d : \mathbb{P}\left[\mathfrak{C}_{\mathbf{x}}^{-\mathbf{u}}\right] > 1 - \alpha\},$$

$$\mathcal{L}_{\mathbf{X}}(\alpha, \mathbf{u}) := \{\mathbf{x} \in \mathbb{R}^d : \mathbb{P}\left[\mathfrak{C}_{\mathbf{x}}^{-\mathbf{u}}\right] < 1 - \alpha\}.$$

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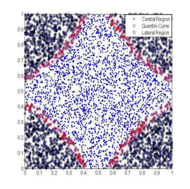
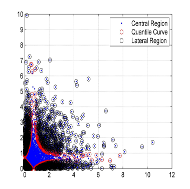
DIRECTIONAL MULTIVARIATE LEVEL-SETS

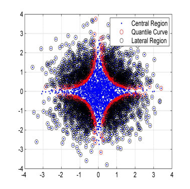
$$\mathbf{u} \in \mathfrak{U} = \left\{ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$$



(A) Bivariate Uniform



(B) Bivariate Exponential

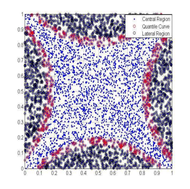


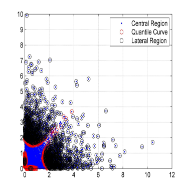
(C) Bivariate Normal

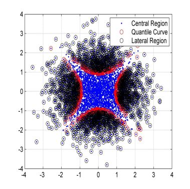
CLASSICAL DIRECTIONS

DIRECTIONAL MULTIVARIATE LEVEL-SETS

$$\mathbf{u} \in \mathfrak{U} = \{(1,0)\,,(0,1)\,,(-1,0)\,,(0,-1)\}$$







(A) Bivariate Uniform (B) Bivariate Exponential (C) Bivariate Normal

CANONICAL DIRECTIONS

Directional Multivariate Quantile Properties

- **Quasi-Odd Property**: $\mathcal{Q}_{-\mathbf{X}}(\alpha, \mathbf{u}) = -\mathcal{Q}_{\mathbf{X}}(\alpha, -\mathbf{u})$.

- **Positive Homogeneity and Translation Invariance**:

$$\mathcal{Q}_{c\mathbf{X}+\mathbf{b}}(\alpha, \mathbf{u}) = c\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) + \mathbf{b}, \text{ for all } c \in \mathbb{R}^{+} \text{ and } \mathbf{b} \in \mathbb{R}^{d}.$$

- **Orthogonal Quasi-Invariance**: Let $\mathbf{w}$ and $\mathbf{u}$ be two unit vectors. Then, an orthogonal matrix Q exists, such that,

$$Q\mathbf{u} = \mathbf{w} \text{ and } \mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) = Q'\mathcal{Q}_{Q\mathbf{X}}(\alpha, \mathbf{w}).$$

Outline

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Review on multivariate out-sample estimation

- **Optimization processes estimation (e.g., Girard and Stupfler (2015))**
- **Estimation of level curves of joint density functions or depth functions (e.g., Cai et al. (2011), Einmahl et al. (2013), He and Einmahl (2017))**
- **Estimation of level curve of either joint distribution or survival functions (e.g. De Haan and Huang (1995))**

Necessary background

Assumption 1, A1.

The random vector $\mathbf{X}$ must be absolutely continuous.

Assumption 2, A2.

Given $\mathbf{u}$, $R_{\mathbf{u}}\,\mathbf{X}$ possesses positive upper-end points of the marginal distributions.

Necessary background

Definition

$\mathbf{X}$ has first order multivariate regular variation with tail index γ, denoted by $RV_{1/\gamma}$, if there exists a real-value function $\phi(t) > 0$ that is regularly varying at infinity with exponent $1/\gamma$ and a non-zero measure $\mu(\cdot)$ on the Borel $\sigma-$field $\bar{\mathbb{R}}^d \backslash \{\mathbf{0}\}$ such that for every Borel set B,

$$t\,\mathbb{P}[(\phi(t))^{-1}\mathbf{X} \in B] \xrightarrow{v} \mu(B),$$

where $\xrightarrow{v}$ means vague convergence and $t \to \infty$.

Assumption 3, A3.

$\mathbf{X}$ has 1st order multivariate regular variation with index γ.

NECESSARY BACKGROUND

DEFINITION

$\mathbf{X}$ has second order multivariate regular variation if there exist functions $\phi(\cdot) \in RV_{1/\gamma}$ and $\Lambda(t) \to 0$, such that $|\Lambda| \in RV_{\pi}$, $\pi \leq 0$; satisfying for all relatively compact rectangles $B \in \bar{\mathbb{R}}^d \backslash \{\mathbf{0}\}$,

$$\frac{t\mathbb{P}\left[(\phi(t))^{-1}\mathbf{X} \in B\right] - \mu(B)}{\Lambda(\phi(t))} \to \psi(B),$$

where $\psi(B)$ is finite and not identically zero.

ASSUMPTION 4, A4.

$\mathbf{X}$ has 2nd order multivariate regular variation with indexes (γ, π).

Directional Results

Result

If $\mathbf{X}$ has 1st(2nd) order multivariate regular variation. Then, $Q\mathbf{X}$ has 1st(2nd) order multivariate regular variation, for any orthogonal transformation Q.

Corollary

If $\mathbf{X}$ has 1st(2nd) order multivariate regular variation. Then the marginals of $Q\mathbf{X}$ has 1st(2nd) order multivariate regular variation.

Characterization of $\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u})$ at high levels

Key tools
A1-A3 and
$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}) = R'_{\mathbf{u}} \mathcal{Q}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e})$

$\mathcal{Q}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e}) \approx \mathcal{Q}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e}, \boldsymbol{\theta})$
where $\boldsymbol{\theta}$ belongs to the unit $d-$dimensional ball and $0 \leq \theta_j \leq 1$

Asymptotic characterization of $\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}, \boldsymbol{\theta})$

$$\begin{aligned}\mathbf{x}_{\mathbf{u}}(\alpha, \boldsymbol{\theta}) &= (x_{\mathbf{u},1}(\alpha, \boldsymbol{\theta}), ..., x_{\mathbf{u},d}(\alpha, \boldsymbol{\theta})) \\ &= \left(a_{\mathbf{u},j}(t) \frac{(\rho_{\mathbf{u}}(\boldsymbol{\theta})\theta_j / t\alpha)^{\gamma} - 1}{\gamma} + b_{\mathbf{u},j}(t); \, j = 1, \ldots, d, \right),\end{aligned}$$

$$\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}, \boldsymbol{\theta}) = R'_{\mathbf{u}} \mathcal{Q}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e}, \boldsymbol{\theta}).$$

How was the characterization possible?

1. Pre-rotation of $\mathbf{X}$ through the orthogonal matrix $R_{\mathbf{u}}$ introduced in the QR orthant definition.

All the elements with subindex $\mathbf{u}$ refer to expressions related to $R_{\mathbf{u}}\mathbf{X}$. For instance, $F_{\mathbf{u}}$ denotes the joint distribution and its marginals are $F_{\mathbf{u},j}$, $j = 1, ..., d$.

2. Key asymptotic result from the Multivariate Extreme Value Theory.

Distribution of convergence of the sample maxima

There exist two sequences $\mathbf{a}_{\mathbf{u}}(\lfloor t \rfloor)$, $\mathbf{b}_{\mathbf{u}}(\lfloor t \rfloor)$ such that,

$$\lim_{t \to \infty} t\left(1 - F_{\mathbf{u}}\left(a_{\mathbf{u},j}(\lfloor t \rfloor)\, x_{\mathbf{u},j} + b_{\mathbf{u},j}(\lfloor t \rfloor);\;\; j = 1, \ldots, d\right)\right) = -\ln\left(G_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}})\right),$$

where $\lfloor \cdot \rfloor$ is the floor function.

HOW WAS THE CHARACTERIZATION POSSIBLE?

3. Marginal high level estimations through extreme value analysis.

HIGH LEVEL QUANTILES OF $F_{\mathbf{u},j}$, $j = 1, \ldots, d$

$$x_{\mathbf{u},j}(\alpha) \approx a_{\mathbf{u},j}(t)\frac{(1/t\alpha)^{\gamma} - 1}{\gamma} + b_{\mathbf{u},j}(t).$$

4. Polar representation in $\mathbb{R}^d$.

POLAR PARAMETRIZATION

In $\mathbf{R}^d$, any point $\mathbf{x}$ can be written in polar coordinates as $\mathbf{x} = ||\mathbf{x}||\,(\mathbf{x}/||\mathbf{x}||) = \rho(\boldsymbol{\theta})\,\boldsymbol{\theta}$, where $\rho(\boldsymbol{\theta}) \in \mathbf{R}^+$ and $\boldsymbol{\theta}$ belonging to the unit $d-$dimensional ball.

How was the characterization possible?

5. Heuristic link between marginal quantiles of $R_{\mathbf{u}}\mathbf{X}$ and $\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u})$ at high levels.

Bivariate ideas found in De Haan and Huang (1995)

The set of solutions to $1 - F(x_1, x_2) = \alpha$ for a bivariate distribution F can be parametrized in polar coordinates as $(\rho(\theta)cos(\theta),\ \rho(\theta)sin(\theta))$, where $\rho(\theta)$ is a solution of the following equations,

$$x_1(\alpha, \theta) = a_1(t)\frac{(\rho(\theta)cos(\theta)/t\alpha)^{\gamma_1} - 1}{\gamma_1} + b_1(t),$$

$$x_2(\alpha, \theta) = a_2(t)\frac{(\rho(\theta)sin(\theta)/t\alpha)^{\gamma_2} - 1}{\gamma_2} + b_2(t).$$

How was the characterization possible?

Dimension 2

Dimension 3

How was the characterization possible?

6. Deduction of the solution for the scalar function $\rho_{\mathbf{u}}$.

Solution in terms of the tail function

Given that,

$$\alpha = 1 - F_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}(\alpha, \boldsymbol{\theta})) \approx t^{-1} \left\{ -\ln \left(G_{\mathbf{u}} \left(\frac{x_{\mathbf{u},j}(\alpha, \boldsymbol{\theta}) - b_{\mathbf{u},j}(t)}{a_{\mathbf{u},j}(t)};\ \ j = 1, \ldots, d \right) \right) \right\}.$$

Then,

$$\rho_{\mathbf{u}}(\boldsymbol{\theta}) := -\ln \left(G_{\mathbf{u}} \left(\frac{\theta_j^{\gamma} - 1}{\gamma};\ \ j = 1, \ldots, d \right) \right).$$

Here $-\ln(G_{\mathbf{u}}(\mathbf{z})$ is the tail function of the multivariate extreme value distribution where the distributions of the multivariate sample maxima converge.

Estimation of $\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u})$ at high levels

Proposed estimator for the elements in $\mathcal{Q}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e}, \boldsymbol{\theta})$

$$\hat{x}_{\mathbf{u},j}(\alpha, \boldsymbol{\theta}, n/k) := \hat{a}_{\mathbf{u},j}(n/k) \left\{ \frac{\left(\frac{k\, \hat{\rho}_{\mathbf{u}}(\boldsymbol{\theta})}{n\, \alpha} \theta_j \right)^{\hat{\gamma}} - 1}{\hat{\gamma}} \right\} + \hat{b}_{\mathbf{u},j}(n/k), \text{ for all } j = 1, \ldots, d.$$

Final $\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}, \boldsymbol{\theta})$ estimator

$$\hat{\mathcal{Q}}_{\mathbf{X}}(\alpha, \mathbf{u}, \boldsymbol{\theta}, n/k) = R'_{\mathbf{u}} \hat{\mathcal{Q}}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e}, \boldsymbol{\theta}, n/k).$$

Estimation of $\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u})$ at high levels

Proposed estimator for the elements in $\mathcal{Q}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e}, \boldsymbol{\theta})$

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Final $\mathcal{Q}_{\mathbf{X}}(\alpha, \mathbf{u}, \boldsymbol{\theta})$ estimator

$$\hat{\mathcal{Q}}_{\mathbf{X}}(\alpha, \mathbf{u}, \boldsymbol{\theta}, n/\textcircled{k}) = R'_{\mathbf{u}} \hat{\mathcal{Q}}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e}, \boldsymbol{\theta}, n/\textcircled{k}).$$

ELEMENTS TO ESTIMATE $Q_{\mathbf{X}}(\alpha, \mathbf{u})$ AT HIGH LEVELS

- **Marginal tail indexes $\hat{\gamma}$ (Dekkers et al. (1989)),**

$$\hat{\gamma} := \mathbf{M}_{k,j}^{(1)} + 1 - \frac{1}{2}\left\{1 - \left(\mathbf{M}_{k,j}^{(1)}\right)^2 / \mathbf{M}_{k,j}^{(2)}\right\}^{-1},$$

where,

$$\mathbf{M}_{k,j}^{(r)} := \frac{1}{k}\sum_{j=0}^{k-1}\{\ln([(R_{\mathbf{u}}\mathbf{X})_j]_{n-i:n}) - \ln([(R_{\mathbf{u}}\mathbf{X})_j]_{n-k:n})\}^r, \quad r = 1, 2.$$

Elements to estimate $Q_{\mathbf{X}}(\alpha, \mathbf{u})$ at high levels

- **The sequences $\hat{\mathbf{a}}_{\mathbf{u}}(n/k)$ and $\hat{\mathbf{b}}_{\mathbf{u}}(n/k)$ (De Haan and Huang (1995)),**

$$\hat{a}_{\mathbf{u},j}(n/k) := [(R_{\mathbf{u}}\mathbf{X})_j]_{n-k:n}\mathbf{M}_{k,j}^{(1)} \max(1, 1-\hat{\gamma}),$$

$$\hat{b}_{\mathbf{u},j}(n/k) := [(R_{\mathbf{u}}\mathbf{X})_j]_{n-k:n}.$$

ELEMENTS TO ESTIMATE $Q_{\mathbf{X}}(\alpha, \mathbf{u})$ AT HIGH LEVELS

- **The scalar function $\hat{\rho}_{\mathbf{u}}(\theta)$. Given that,**

$$-\ln\left(G_{\mathbf{u}}\left(\mathbf{x}_{\mathbf{u}}\right)\right) \approx \frac{n}{k}\left[1 - F_{\mathbf{u}}\left(\frac{x_{\mathbf{u},j} - b_{\mathbf{u},j}(n/k)}{a_{\mathbf{u},j}(n/k)};\ \ j = 1, \ldots, d\right)\right]$$

Then,

$$\hat{\rho}_{\mathbf{u}}(\boldsymbol{\theta}) := \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}_{\left\{\bigcup_{j=1}^{d}\left[(R_{\mathbf{u}}\mathbf{X}_i)_j > \hat{a}_{\mathbf{u},j}(n/k)x_{\mathbf{u},j} + \hat{b}_{\mathbf{u},j}(n/k)\right]\right\}}$$

BOOTSTRAP-BASED METHOD TO SELECT $k(n)$

The multivariate tuning parameter selection based on the univariate procedure by Danielsson et al. (2001).

ORTHANT ORDER IN (TORRES ET AL. 2015)

$\mathbf{x}$ is said to be less than $\mathbf{y}$ in direction $\mathbf{u}$ if:

$$\mathbf{x} \preceq_{\mathbf{u}} \mathbf{y} \quad \equiv \quad \mathfrak{C}_{\mathbf{x}}^{\mathbf{u}} \supseteq \mathfrak{C}_{\mathbf{y}}^{\mathbf{u}} \quad \equiv \quad R_{\mathbf{u}}\mathbf{x} \leq R_{\mathbf{u}}\mathbf{y}.$$

- STEP 1. Pre-rotate the sample to generate $\{R_{\mathbf{u}}\mathbf{x}_1, \ldots, R_{\mathbf{u}}\mathbf{x}_n\}$ and center that with respect to its mean.
- STEP 2. Set $m_1 = \lfloor n^{1-\epsilon} \rfloor$ for some $\epsilon \in (0, 1/2)$. Draw a large number B_1 of bootstrap samples of size m_1 and order each of them according to the orthant order, dropping the observations with non-positive components.

BOOTSTRAP-BASED METHOD TO SELECT $k(n)$

STEP 3. Denote the bootstrap error of each marginal $j = 1, \ldots, d$ by,

$$Err_j(m_1, b_1, k_j) := \left(\mathbf{M}_{k_j,j}^{(2)} - 2\left(\mathbf{M}_{k_j,j}^{(1)}\right)^2\right)^2, \quad b_1 = 1, \ldots, B_1,$$

where k_j varies from 1 to $m_1 - 1$. Then, determine the value $k_j(m_1)$ that minimizes the mean sample error,

$$\frac{1}{B_1} \sum_{b_1=1}^{B_1} Err_j(m_1, b_1, k_j).$$

Bootstrap-based method to select $k(n)$

- STEP 4. Set $m_2 = \lfloor m_1^2/n \rfloor$, and repeat Step 2 to obtain $k_j(m_2)$.
- STEP 5. Estimate the associated second order tail index π by

$$\hat{\pi} = \frac{1}{d}\sum_{j=1}^{d} \log\left(\frac{k_j(m_1)}{-2\log(m_1) + 2\log(k_j(m_1))}\right),$$

  which is a consistent estimator, (see Qi (2008)).
- STEP 6. The optimal selection for $k = k(n)$ is given by,

$$\hat{k}(n) := \frac{1}{d}\sum_{j=1}^{d} \frac{k_j(m_1)^2}{k_j(m_2)}\left(1 - \frac{1}{\hat{\pi}}\right)^{1/(2\hat{\pi}-1)}.$$

Asymptotic Normality for $\hat{Q}_{\mathbf{X}}(\alpha, \mathbf{u})$ at high levels

Result

If $R_{\mathbf{u}}\mathbf{X}$ is a second order multivariate regularly varying random vector. Then,

$$\sqrt{k}\left(\frac{\hat{x}_{\mathbf{u},j}(\alpha,\boldsymbol{\theta}) - x_{\mathbf{u},j}(\alpha,\boldsymbol{\theta})}{\hat{a}_{\mathbf{u},j}(n/k)\int_1^{s_n} t^{\hat{\gamma}-1}(\log t)dt};\ \ j = 1,\ldots,d\right),$$

converges to a multivariate normal distribution.

Corollary

The asymptotic normality of the elements in $\hat{Q}_{R_{\mathbf{u}}\mathbf{X}}(\alpha, \mathbf{e}, \boldsymbol{\theta})$ implies the asymptotic normality of the elements in $\hat{Q}_{\mathbf{X}}(\alpha, \mathbf{u}, \boldsymbol{\theta})$.

ILLUSTRATIVE EXAMPLE $\equiv t-$DISTRIBUTION

If $\mathbf{X}$ has a multivariate $t-$distribution, then holds A1-A4 and this distribution is closed under orthogonal transformations.

Original Space

$$\boldsymbol{\mu}$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{pmatrix}$$

$$\nu$$

Rotated Space in direction $\mathbf{u}$

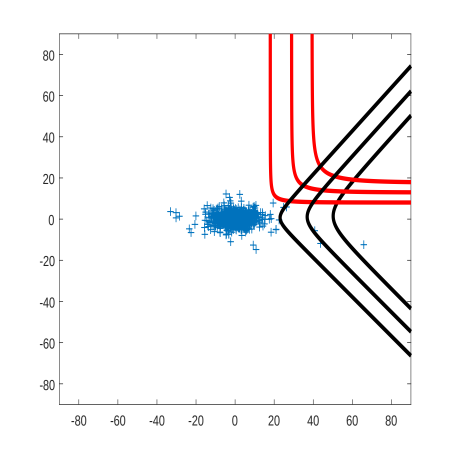
$$\boldsymbol{\mu}_{\mathbf{u}} = R_{\mathbf{u}}\boldsymbol{\mu}$$

$$\Sigma_{\mathbf{u}} = R_{\mathbf{u}}\Sigma R_{\mathbf{u}}'$$

$$\nu$$

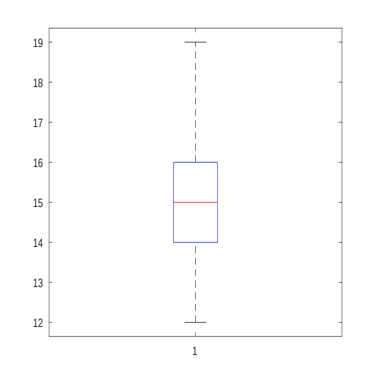
$$\text{Example} \equiv \left\{ \begin{matrix} \boldsymbol{\mu} = (0,\, 0)',\, \sigma_1^2 = 5,\, \sigma_2^2 = 1,\, \sigma_{1,2} = 0.1,\, \nu = 3 \\ \alpha = 1/n, \mathbf{u} = \mathbf{e},\, \textbf{FPC} \end{matrix} \right\}$$

Directional Multivariate Quantiles

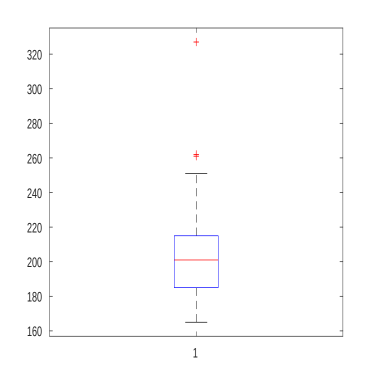


Theoretical *DMQ* for $\alpha \in \{0.5, 0.3, 0.1\}$ and $\mathbf{u} \in \{\mathbf{e}, \textbf{FPC}\}$

BOOTSTRAP-BASED DISTRIBUTION OF k

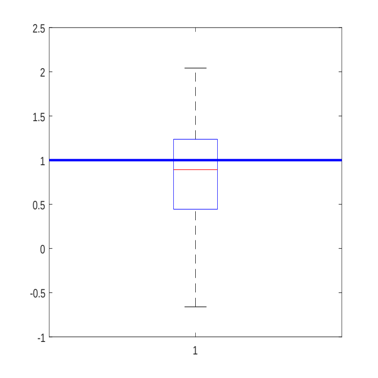


(A) $n = 500, \quad \alpha = \frac{1}{n}$



(B) $n = 5000, \quad \alpha = \frac{1}{n}$

Boxplots of the ratios $\hat{\gamma}_{\mathbf{u},1}/\gamma_{\mathbf{u},1}$



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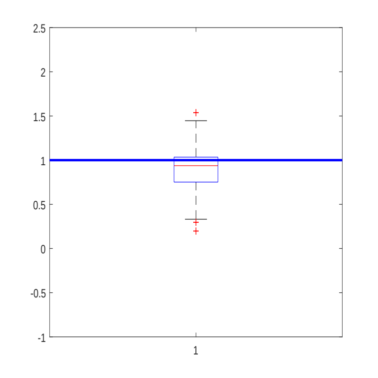
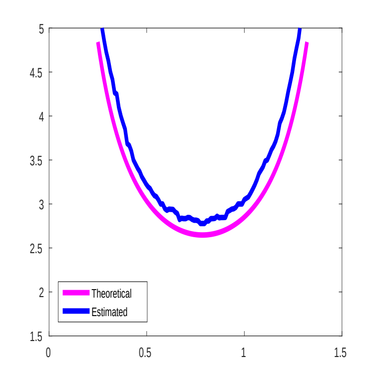


(B) $n = 5000, \quad \alpha = \frac{1}{n}$

Theoretical value $\gamma = 1/\nu, j = 1, 2$

ESTIMATION OF $\rho_{\mathbf{u}}$



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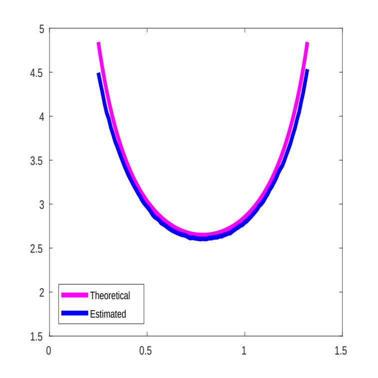
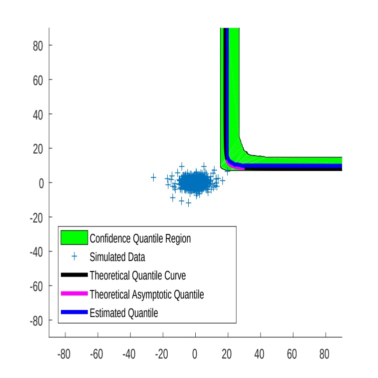


(B) $n = 5000, \quad \alpha = \frac{1}{n}$

Theoretical expression by Nikoloulopoulos et al. (2009)

FINAL ESTIMATION IN THE CLASSICAL DIRECTION e



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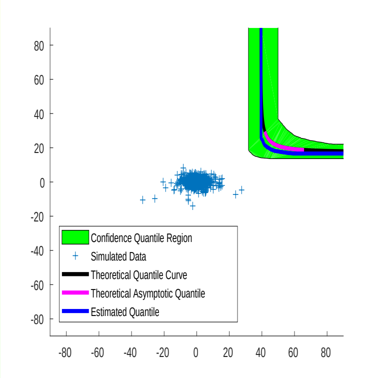
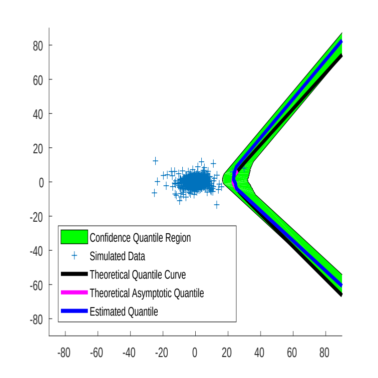
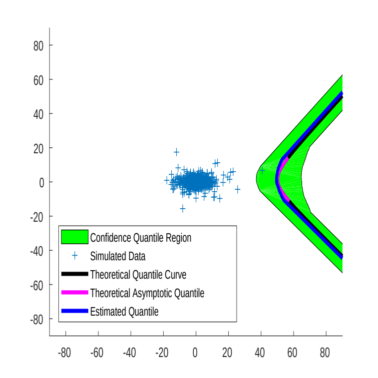


(B) $n = 5000, \quad \alpha = \frac{1}{n}$

Estimation in the FPC direction



(A) $n = 500, \quad \alpha = \frac{1}{n}$



(B) $n = 5000, \quad \alpha = \frac{1}{n}$

3D EXAMPLE

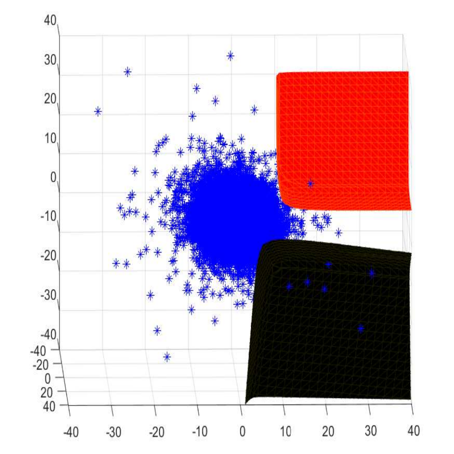
Parameters of the $t-$distribution

$$\boldsymbol{\mu} = (0,\, 0,\, 0)'$$

$$\Sigma = \begin{pmatrix} 5 & 2.44 & -1.88 \\ 2.44 & 2.12 & 0.04 \\ -1.88 & 0.04 & 2.36 \end{pmatrix}$$

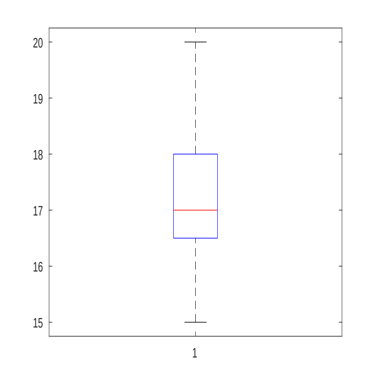
$$\nu = 4$$

DIRECTIONAL MULTIVARIATE QUANTILES

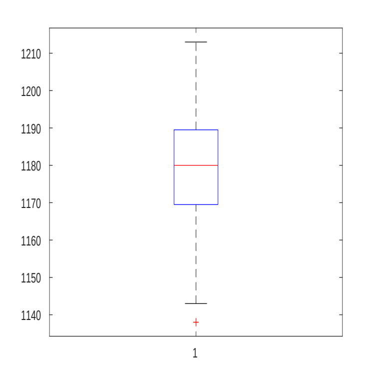


Theoretical DMQ for $\alpha = 0.1$ and $\mathbf{u} \in \{\mathbf{e}, \textbf{FPC}\}$

BOOTSTRAP-BASED DISTRIBUTION OF k

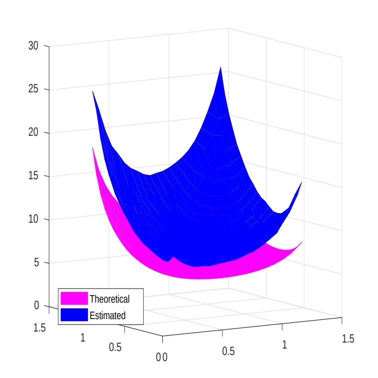


(A) $n = 500, \quad \alpha = \frac{1}{n}$

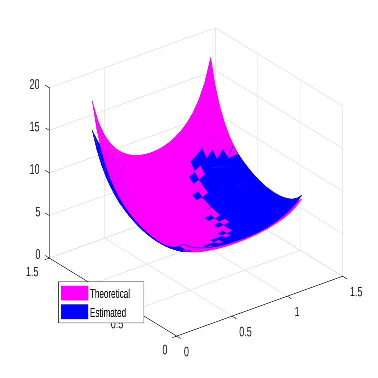


(B) $n = 50000, \quad \alpha = \frac{1}{n}$

ESTIMATION OF $\rho_{\mathbf{u}}$



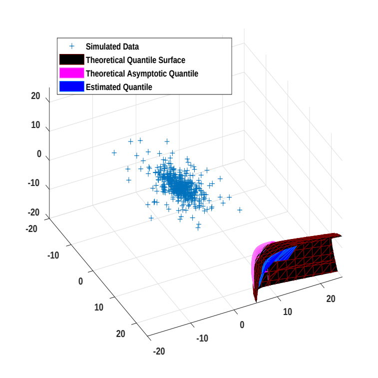
(A) $n = 500, \quad \alpha = \frac{1}{n}$



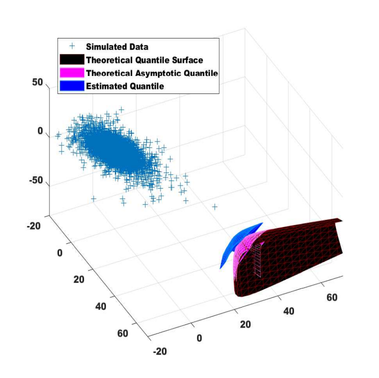
(B) $n = 50000, \quad \alpha = \frac{1}{n}$

Theoretical expression by Nikoloulopoulos et al. (2009)

ESTIMATION IN THE FPC DIRECTION

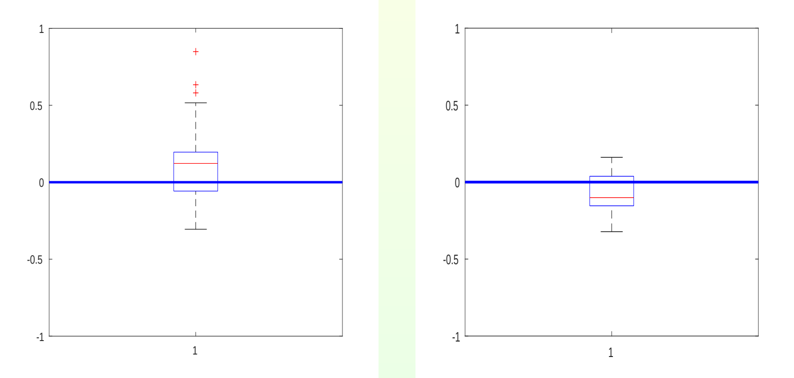


(A) $n = 500, \quad \alpha = \frac{1}{n}$



(B) $n = 50000, \quad \alpha = \frac{1}{n}$

Relative error of $\hat{\mathcal{Q}}_{\mathbf{X}}(\alpha, \mathbf{u}, \boldsymbol{\theta}, n/k)$, where $\boldsymbol{\theta} = (1/\sqrt{d}, \ldots, 1/\sqrt{d})$



(A) $n = 500, \quad \alpha = \frac{1}{n}$ (B) $n = 50000, \quad \alpha = \frac{1}{n}$

Outline

1. Directional Notions
2. Non-parametric Out-sample Estimation
3. Real case study
4. Conclusions

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REAL CASE STUDY

Portfolio of Indices	$\equiv$	(S&P 500, FTSE 100, Nikkei 225) (USA, UK, Japan)

Data from July 2nd, 2001 to June 29th, 2007

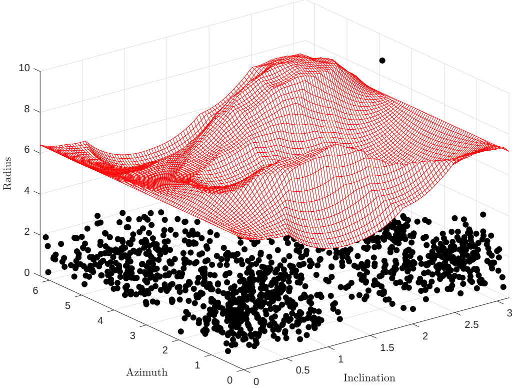
↓

GARCH modeling to ensure i.i.d.

↓

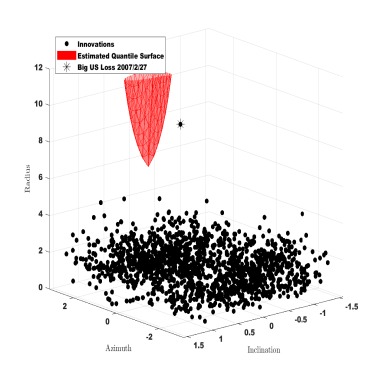
Directional Analysis of the Filtered Losses

Overall analysis

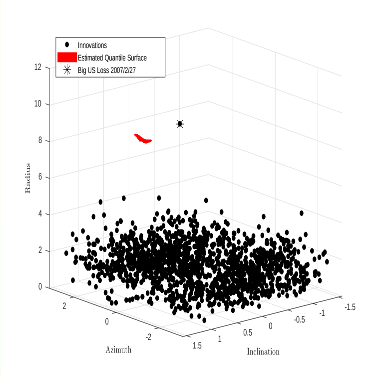


Outlier criteria through Tukey depth trimming, $\alpha = 1/10000$, (He and Einhmal (2017))

EVEN INVESTMENT (WEIGHTS = $1/\sqrt{3}(1,1,1)$, LEVERAGE BUT NOT OUTLIER)



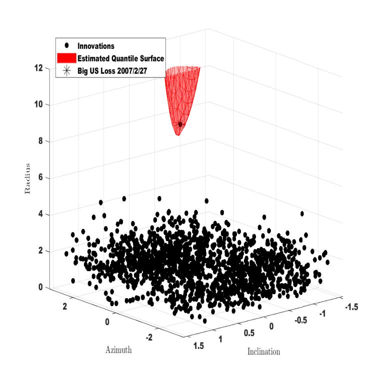
(A) Semi-parametric approach



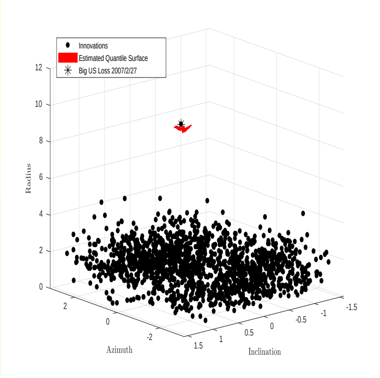
(B) Non-parametric approach

Directional portfolio criteria, $\mathbf{u} = \mathbf{e}$ and $\alpha = 1/1250$.

A Critical Investment (weights = (0.6,0.35,0.05), identified outlier)



(A) Semi-parametric approach



(B) Non-parametric approach

Directional portfolio criteria, $\mathbf{u} = (0.6, 0.35, 0.05)$ and $\alpha = 1/1250$.

Outline

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CONCLUSIONS

- **Results that plug the directional approach into the multivariate value theory have been proved.**
- **A non-parametric procedure to perform *out-sample* estimation of the directional multivariate quantiles has been developed.**
- **A bootstrap-based method of selection for the tuning parameter k has been introduced.**
- **The asymptotic normality of the estimator has been shown.**
- **The performance of the estimation at high levels has been shown in a heavy tailed example.**
- **A real case study of a decision rule to determine the existence of an outlier has been shown.**

Thanks

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Thanks